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A formal moduli space of symplectic connections of Ricci-type on T^{2n}

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Abstract

We consider analytic curves ∇^t of symplectic connections of Ricci-type on the torus T^{2n} with ∇^0 the standard connection. We show, by a recursion argument, that if ∇^t is a formal curve of such connections then there exists a formal curve of symplectomorphisms ψ_t such that $\psi_t \cdot \nabla^t$ is a formal curve of flat T^{2n}-invariant symplectic connections and so ∇^t is flat for all t. Applying this result to the Taylor series of the analytic curve, it means that analytic curves of symplectic connections of Ricci-type starting at ∇^0 are also flat.

The group G of symplectomorphisms of the torus (T^{2n}, ω) acts on the space $\mathcal{E}$ of symplectic connections which are of Ricci-type. As a preliminary to study the moduli space $\mathcal{E}/G$ we study the moduli of formal curves of connections under the action of formal curves of symplectomorphisms.

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1. Introduction

On any symplectic manifold (M, ω) the space $\mathcal{S}$ of symplectic connections is an infinite dimensional affine space whose corresponding vector space is the space of completely

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symmetric 3-tensors on M. To encode some geometry into a symplectic connection it thus seems reasonable to introduce a selection rule for symplectic connections. A variational principle associated to a Lagrangian density, which is an invariant quadratic polynomial in the curvature, has been considered in [1]; the symplectic connections satisfying the Euler–Lagrange equations are said to be *preferred*. The symplectomorphism group G of (M, ω) acts naturally on $\mathcal{S}$ and stabilises the subspace $\mathcal{P}$ of preferred symplectic connections. The first question we wanted to address is to give a description of the moduli space $\mathcal{P}/G$ of preferred connections modulo the action of symplectomorphisms. Such a description was given in [1] when (M, ω) is a closed surface; but, up to now, very little has been done in the higher-dimensional situation.

We have observed that a linear condition on the curvature (the vanishing of one of its irreducible components—the non-Ricci component, W) implies the Euler–Lagrange equations. Furthermore, this condition seems to imply that many of the properties of the surface situation extend to the higher-dimensional case. We have called symplectic connections satisfying this curvature condition *connections of Ricci-type* (all symplectic connections in dimension 2 are of Ricci-type). This condition is preserved by symplectomorphisms and so we modify our initial question to the following one: give a description of the space $\mathcal{E}$ of Ricci-type connections and its moduli space $\mathcal{E}/G$.

This paper is devoted to this modified question in the case where M is a torus T^{2n} and ω a T^{2n}-invariant symplectic structure. Although we do not answer this question, we are able, in a formal setting made precise below, to show that the moduli space is infinite dimensional and to give a partial description of it.

If ∇^t is a formal curve of symplectic connections, we shall denote by W^t the W part of the curvature of ∇^t. We prove the following:

Theorem. *Let ∇^t be a formal curve of symplectic connections on (T^{2n}, ω) such that ∇^0 is the standard flat connection on T^{2n}, and such that $W^t = 0$. Then the formal curvature R^t of ∇^t vanishes and there exists a formal curve of symplectomorphisms ψ_t such that $\tilde{\nabla}^t := \psi_t \cdot \nabla^t$ is a formal curve of flat T^{2n}-invariant symplectic connections.*

This implies the following:

Theorem. *Let ∇^t be an analytic curve of analytic symplectic connections on (T^{2n}, ω) such that ∇^0 is the standard flat connection on T^{2n}, and such that $W^t = 0$. Then the curvature R^t of ∇^t vanishes.*

For the moduli space in the formal setting, we show:

Proposition. *For two curves $\tilde{\nabla}^t$ and $\tilde{\nabla}'^t$ of invariant flat connections of Ricci-type on $(\mathbb{R}^{2n}, \Omega)$ with $\tilde{\nabla}^0 = \tilde{\nabla}'^0$ the trivial connection, there always exists a formal curve of symplectomorphisms $\tilde{\psi}_t$ so that $\tilde{\psi}_t \cdot \tilde{\nabla}^t = \tilde{\nabla}'^t$.*

Theorem. *The moduli space of formal curves of Ricci-type symplectic connections starting with the standard flat connection on (T^{2n}, ω) under the action of formal curves of symplectomorphisms is described by the space of formal curves of linear maps $A^t : \mathbb{R}^{2n} \to$*

$\mathfrak{sp}(2n, \mathbb{R})[\![t]\!]$ *satisfying* $A^t(X)A^t(Y) = 0$ *and* $A^t(X)Y = A^t(Y)X$, *modulo the action of* $Sp(2n, \mathbb{Z})$.

The plan of the paper is as follows. In Section 2 we recall some general properties of symplectic connections having Ricci-type curvature. In Section 3 we introduce the notion of formal curves of connections and we show that the properties of Section 2 are still true for a formal curve of symplectic connections with Ricci-type curvature. In Section 4, we analyse the $W^t = 0$ condition at order 1 and 2 for $\nabla^t = \nabla^0 + \sum_{k=1}^{\infty} t^k A^{(k)}$ a formal curve of Ricci-type symplectic connections on T^{2n} with ∇^0 the standard flat connection; in particular, we show that there exists a function $U^{(1)}$ and a completely symmetric, T^{2n}-invariant 3-tensor $Q^{(1)}$ on T^{2n} such that $\underline{A}^{(1)} = (\nabla^0)^3 U^{(1)} + Q^{(1)}$ and we show that $\nabla'^t = \nabla^0 + t\bar{Q}^{(1)}$ (with $\omega(\bar{Q}^{(1)}(X)Y, Z) = Q^{(1)}(X, Y, Z)$) defines a curve of invariant flat symplectic connections on (T^{2n}, ω). This remark can be formulated in a slightly different way: given $\nabla^t = \nabla^0 + A^{(t)}$ a smooth curve of Ricci-type symplectic connections then, up to a symplectomorphism, the tangent vector to this family of connections lies in the finite dimensional space of flat T^{2n}-invariant symplectic connections. Section 5 is devoted to a proof of a recurrence lemma which implies the first theorem. In Section 6 we study the question of when two formal curves of flat invariant connections on T^{2n} are equivalent by a formal curve of symplectomorphisms.

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2. Ricci-type curvature

A symplectic connection ∇ on a symplectic manifold (M, ω) is a linear connection having no torsion and for which ω is parallel ($\nabla\omega = 0$). The curvature endomorphism R of ∇ is

$$R(X, Y)Z = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z$$

for vector fields X, Y, Z on M. The symplectic curvature tensor

$$R(X, Y; Z, T) = \omega(R(X, Y)Z, T)$$

is antisymmetric in its first two arguments, symmetric in its last two and satisfies the first Bianchi identity

$$\underset{X,Y,Z}{\circlearrowleft\!\!\!\!\!\!\bigoplus} R(X, Y; Z, T) = 0,$$

where $\underset{bcd}{\circlearrowleft\!\!\!\!\!\!\bigoplus}$ denotes the sum over the cyclic permutations of the listed set of elements. The second Bianchi identity takes the form

$$\underset{X,Y,Z}{\circlearrowleft\!\!\!\!\!\!\bigoplus} (\nabla_X R)(Y, Z) = 0.$$

The Ricci tensor r is the symmetric 2-tensor

$$r(X, Y) = \mathrm{Trace}\,[Z \mapsto R(X, Z)Y].$$

If dim $M = 2n \geq 4$, the curvature R of such a connection has two irreducible components under the action of the symplectic group $Sp(2n, \mathbb{R})$. We denote them by E and W:

$$R = E + W.$$

The E component encodes the information contained in the Ricci tensor of ∇ and is called the Ricci part of the curvature tensor. It is given by

$$E(X, Y; Z, T) = \frac{-1}{2(n+1)}[2\omega(X, Y)r(Z, T) + \omega(X, Z)r(Y, T) + \omega(X, T)r(Y, Z) - \omega(Y, Z)r(X, T) - \omega(Y, T)r(X, Z)].$$

The curvature is said to be Ricci-type if the W component vanishes, i.e. when $R = E$.

Lemma 1. *Let (M, ω) be a symplectic manifold of dimension $2n \geq 4$. If the curvature of a symplectic connection ∇ on M is of Ricci-type then there is a 1-form u such that*

$$(\nabla_X r)(Y, Z) = \frac{1}{2n+1}(\omega(X, Y)u(Z) + \omega(X, Z)u(Y)).$$

Conversely, if there is such a 1-form u, the "Weyl" part of the curvature, $W = R - E$ satisfies

$$\underset{X,Y,Z}{\circlearrowright\!\!\!\!\!\!\oplus}(\nabla_X W)(Y, Z; T, U) = 0.$$

Proof. The property follows from the second Bianchi's identity, see [2]. □

Corollary 2. *A symplectic manifold with a symplectic connection whose curvature is of Ricci-type is locally symmetric if and only if the 1-form u, defined in the lemma, vanishes.*

Denote by ρ the linear endomorphism such that

$$r(X, Y) = \omega(X, \rho Y).$$

The symmetry of r is equivalent to saying that ρ is in the Lie algebra of the symplectic group $Sp(TM, \omega)$. For an integer $p > 1$, define

$$\overset{(p)}{r}(X, Y) = \omega(X, \rho^p Y).$$

It is symmetric when p is odd and antisymmetric when p is even.

Lemma 3. *Let (M, ω) be a symplectic manifold with a symplectic connection ∇ with Ricci-type curvature. Then, the following identities hold:*

(1) *There is a function b such that*

$$\nabla u = -\frac{1+2n}{2(1+n)}\overset{(2)}{r} + b\omega,$$

(2) *The differential of the function b is given by*

$$\mathrm{d}b = \frac{1}{1+n} i(\bar{u})r,$$

where $\bar{u}$ is the vector field such that $i(\bar{u})\omega = u$;

(3) *When M is connected*

$$b + \frac{2n+1}{4(1+n)} \mathrm{Trace}\, \rho^2$$

is constant.

Proof. These identities follow from Lemma 1, see [2]. □

Let the torus T^{2n} be endowed with a T^{2n}-invariant symplectic structure ω. Let ∇ be a symplectic connection on (T^{2n}, ω) which is of Ricci-type. The group G of symplectomorphisms of (T^{2n}, ω) acts on the set $\mathcal{E}$ of symplectic connections with $W = 0$. We are interested in the set of orbits of G in $\mathcal{E}$, i.e. in $\mathcal{E}/G$.

We now consider the symplectic vector space $(\mathbb{R}^{2n}, \Omega)$ and view Ω as a translation invariant symplectic structure. A symplectic connection on $\mathbb{R}^{2n}$ will be determined by its values on translation invariant vector fields. If, in addition, the connection ∇ is translation invariant then $B(X)Y := \nabla_X Y$ (for invariant vector fields X, Y) defines a linear map $B : \mathbb{R}^{2n} \to \mathfrak{sp}(2n, \mathbb{R})$ which completely determines ∇. The only condition on B is that $\Omega(B(X)Y, Z)$ is completely symmetric.

Proposition 4. *Let ∇ be a translation invariant symplectic connection on $(\mathbb{R}^{2n}, \Omega)$ and let $B(X)Y = \nabla_X Y$ as above. If ∇ is of Ricci-type and $2n \geq 4$, then ∇ is flat and $B(X)B(Y) = 0$.*

Proof. Since B is constant, the curvature endomorphism is given by

$$R(X, Y) = [B(X), B(Y)]$$

and so the Ricci tensor is given by

$$r(X, Y) = \mathrm{Trace}\,(B(X)B(Y)).$$

It is easy to see that symplectic curvature tensors $R(X, Y; Z, T)$ are, in fact, determined by the terms of the form $R(X, Y; X, Y)$ so that the equation $W = 0$ is equivalent to $R(X, Y; X, Y) = -(2/(n+1))\Omega(X, Y)r(X, Y)$ and in the present case this has the form

$$(n+1)\Omega(B(X)X, B(Y)Y) = -2\Omega(X, Y)r(X, Y).$$

Polarising the equation in X we have

$$\begin{aligned}(n+1)\Omega(T, B(X)B(Y)Y) &= \Omega(X, Y)r(T, Y) + \Omega(T, Y)r(X, Y) \\ &= \Omega(X, Y)\Omega(T, \rho Y) + \Omega(T, Y)\Omega(X, \rho Y),\end{aligned}$$

so that $W = 0$ is equivalent to

$$(n+1)B(X)B(Y)Y = \Omega(X, Y)\rho Y + \Omega(X, \rho Y)Y.$$

Polarising this in Y we have

$$2(n+1)B(X)B(Y)Z = \Omega(X, Y)\rho Z + \Omega(X, \rho Y)Z + \Omega(X, Z)\rho Y + \Omega(X, \rho Z)Y. \quad (1)$$

Now choose dual bases X^i, X_i for $\mathbb{R}^{2n}$ with $\Omega(X^i, X_j) = \delta^i_j$ then an easy calculation shows

$$\rho = \sum_i B(X^i)B(X_i).$$

If we multiply (1) by $B(X^i)$, set $X = X_i$ and sum we get

$$(n+1)\rho B(Y)Z = -B(Y)\rho Z - B(Z)\rho Y.$$

Alternatively, we may substitute $B(X_i)Z$ for Z in (1), set $Y = X^i$ and sum to give

$$(n+1)B(X)\rho Z = -\rho B(Z)X + B(Z)\rho X.$$

Adding the two equations after setting $X = Y$ we see that

$$\rho B(X) = -B(X)\rho$$

and hence that

$$(n-1)\rho B(X) = 0.$$

Thus, if $2n \geq 4$

$$\rho B(X) = B(X)\rho = 0 \quad \Rightarrow \rho^2 = 0.$$

Substituting ρZ for Z in (1) we have

$$0 = r(X, Y)\rho Z + r(X, Z)\rho Y$$

and setting $Z = Y$, applying $\Omega(X, .)$ we get finally

$$0 = r(X, Y)^2.$$

Thus the Ricci tensor vanishes, and hence ∇ is flat.

Putting $\rho = 0$ in (1) yields $B(X)B(Y) = 0$. □

3. Formal curves

Definition 5. *A formal curve of symplectic connections* on a symplectic manifold (M, ω) is a formal power series

$$\nabla^t = \nabla + \sum_{k=1}^{\infty} t^k A^{(k)},$$

where ∇ is a symplectic connection on M, and the $A^{(k)}$ are (2, 1) tensors such that

$$\underline{A}^{(k)}(X, Y, Z) := \omega(A^{(k)}(X)Y, Z) \quad (2)$$

is totally symmetric.

Definition 6. *A formal curve of symplectomorphisms* is a homomorphism of Poisson algebras

$$\psi_t : C^\infty(M) \to C^\infty(M)[\![t]\!], \qquad \psi_t = \psi^{(0)} + \sum_{k=1}^{\infty} t^k \psi^{(k)}$$

such that $\psi^{(0)} : C^\infty(M) \to C^\infty(M)$ is an isomorphism.

The leading term $\psi^{(0)}$ of a formal curve of symplectomorphisms is given by composition with a symplectomorphism $\psi^{(0)}(f) = f \circ \sigma = \sigma^*(f)$ so that we may take such a term out as a common factor and write $\psi_t = \sigma^* \circ \phi_t$ and $\phi_t = \mathrm{id} + \sum_{k\geq 1} t^k \phi^{(k)}$.

If $\phi_t = \mathrm{id} + \sum_{k\geq 1} t^k \phi^{(k)}$ is a formal curve of symplectomorphisms beginning with the identity then the first-order term $X^{(1)} = \phi^{(1)}$ is a symplectic vector field. Moreover, for any symplectic vector field, $\exp tX = \mathrm{id} + \sum_{k\geq 1} t^k/k! X^k$ is a formal curve of symplectomorphisms. A straightforward recursion argument then shows that any formal curve of symplectomorphisms beginning with the identity can be written in the form $\phi_t = \exp X_t$, where $X_t = \sum_{k\geq 1} t^k X^{(k)}$ is a formal curve of vector fields.

Definition 7. *A formal* 1*-parameter group of symplectomorphisms* is a formal curve of symplectomorphisms ψ_t such that $\psi_{at} \circ \psi_{bt} = \psi_{(a+b)t}$ for all $a, b \in \mathbb{R}$.

In order for this definition to make sense we first have to extend ψ_t by linearity over $\mathbb{R}[\![t]\!]$ to a morphism of $\mathbb{R}[\![t]\!]$ algebras. The definition then implies that $\psi^{(0)}$ is the identity and that $\psi^{(1)}(f) = X(f)$ for some symplectic vector field which we call *the infinitesimal generator* of ψ_t. It is easy to see that every formal 1-parameter group of symplectomorphisms has the form $\psi_t = \exp tX$. Moreover, a recursion shows that, if X_t is a formal curve of symplectic vector fields, we can find a second sequence of symplectic vector fields $Y^{(k)}$ such that

$$\exp X_t = \exp tY^{(1)} \circ \exp t^2 Y^{(2)} \circ \cdots \circ \exp t^k Y^{(k)} \circ \cdots$$

and so any formal curve of symplectomorphisms ψ_t can be factorised in two ways

$$\psi_t = \sigma^* \circ \exp X_t = \sigma^* \circ \phi^{(1)}_t \circ \phi^{(2)}_{t^2} \circ \cdots \circ \phi^{(k)}_{t^k} \circ \cdots ,$$

where the $\phi^{(k)}_t$ are formal 1-parameter groups of symplectomorphisms.

Remark that a formal curve of symplectomorphisms ψ_t acts on a formal curve of vector fields X_t viewed as a $\mathbb{R}[\![t]\!]$-linear derivation of $C^\infty(M)[\![t]\!]$ by

$$(\psi_t \cdot X_t) f = \psi_t(X_t(\psi_t^{-1} f)),$$

and acts on a formal curve of symplectic connections ∇^t by

$$(\psi_t \cdot \nabla^t)_X Y = \psi_t \cdot (\nabla^t_{\psi_t^{-1}\cdot X} \psi_t^{-1} \cdot Y). \tag{3}$$

Let ∇^t be a formal curve of symplectic connections on a symplectic manifold (M, ω) of dimension $2n$,

$$\nabla^t = \nabla + \sum_{k=1}^{\infty} t^k A^{(k)}.$$

We denote as in (2) by $\underline{A}^{(k)}$ the corresponding symmetric 3-tensors. The formal curvature endomorphism R^t of ∇^t is $R^t(X, Y) = \nabla^t_X \circ \nabla^t_Y - \nabla^t_Y \circ \nabla^t_X - \nabla^t_{[X,Y]}$ so that

$$R^t = R^\nabla + \sum_{k=1}^{\infty} t^k R^{(k)}$$

with

$$R^{(k)}(X, Y) = (\nabla_X A^{(k)})(Y) - (\nabla_Y A^{(k)})(X) + \sum_{\substack{p+q=k \\ p,q\geq 1}} [A^{(p)}(X), A^{(q)}(Y)]. \tag{4}$$

The symplectic curvature tensor $R^t(X, Y; Z, T) = \omega(R^t(X, Y)Z, T)$ is antisymmetric in its first two arguments, symmetric in its last two, satisfies the first Bianchi identity $\oint_{X,Y,Z} R^t(X, Y; Z, T) = 0$ and the second Bianchi identity $\oint_{X,Y,Z} (\nabla^t_X R^t)(Y, Z) = 0$.

The formal Ricci tensor is $r^t(X, Y) = \mathrm{Trace}[Z \mapsto R^t(X, Z)Y]$, so that

$$r^t = r^\nabla + \sum_{k=1}^{\infty} t^k r^{(k)},$$

where the $r^{(k)}$ are the symmetric tensors

$$r^{(k)}(X, Y) = \mathrm{Trace}[Z \mapsto (\nabla_Z A^{(k)})(X)Y] + \sum_{\substack{p+q=k \\ p,q\geq 1}} \mathrm{Trace}\, A^{(p)}(X)A^{(q)}(Y). \tag{5}$$

The Ricci part E^t of the formal curvature tensor is given by

$$\begin{aligned} E^t(X, Y; Z, T) = \frac{-1}{2(n+1)}[&2\omega(X, Y)r^t(Z, T) + \omega(X, Z)r^t(Y, T) + \omega(X, T)r^t(Y, Z) \\ &-\omega(Y, Z)r^t(X, T) - \omega(Y, T)r^t(X, Z)]. \end{aligned} \tag{6}$$

The formal curvature is said to be of Ricci-type when $R^t = E^t$.

Lemma 8. *Let (M, ω) be a symplectic manifold of dimension $2n \geq 4$. If the formal curvature of a formal curve of symplectic connections ∇^t on M is of Ricci-type then there exists a formal curve of 1-forms*

$$u^t = \sum_{k=0}^{\infty} t^k u^{(k)}$$

such that

$$(\nabla^t_X r^t)(Y, Z) = \frac{1}{2n+1}(\omega(X, Y)u^t(Z) + \omega(X, Z)u^t(Y)) \tag{7}$$

and there exists a formal curve of functions

$$b^t = \sum_{k=0}^{\infty} t^k b^{(k)}$$

such that

$$\nabla^t u^t = -\frac{1+2n}{2(1+n)} \overset{(2)}{r^t} + b^t \omega. \tag{8}$$

with $\omega(X, (\rho^t)Y) = r^t(X, Y) =$ *and* $r^{t\,(2)}(X, Y) = \omega(X, (\rho^t)^2 Y)$. *Also*

$$\mathrm{d}b^t = \frac{1}{1+n} i(\bar{u}^t) r^t. \tag{9}$$

Lemma 9. *Let* ∇^t *be a formal curve of translation invariant symplectic connections on* $(\mathbb{R}^{2n}, \Omega)$ *and let* $B^t(X)Y := \nabla^t_X Y$ *(for invariant vector fields* X, Y*). If* ∇^t *is of Ricci-type and* $2n \geq 4$*, then* ∇^t *is flat and* $B^t(X)B^t(Y) = 0$.

Proof. We can copy in the formal series setting the proof of Lemma 9. Write $B^t = \sum_{k=0}^{\infty} t^k B^{(k)}$ where the $B^{(k)}$ are constant maps from $\mathbb{R}^{2n}$ to $sp(\mathbb{R}^{2n}, \Omega)$. The formal curvature endomorphism is given by

$$R^t(X, Y) = [B^t(X), B^t(Y)],$$

i.e.

$$R^{(k)}(X, Y) = \sum_{\substack{p+q=k\\ p,q\geq 0}} [B^p(X), B^q(Y)]$$

and the formal Ricci tensor is given by

$$r^t(X, Y) = \mathrm{Trace}(B^t(X)B^t(Y)),$$

i.e.

$$r^{(k)}(X, Y) = \sum_{\substack{p+q=k\\ p,q\geq 0}} \mathrm{Trace}\, B^p(X)B^q(Y).$$

The equation $W^t = 0$ is again equivalent to

$$2(n+1)B^t(X)B^t(Y)Z = \Omega(X, Y)\rho^t Z + \Omega(X, \rho^t Y)Z + \Omega(X, Z)\rho^t Y + \Omega(X, \rho^t Z)Y,$$

i.e.

$$\sum_{\substack{p+q=k\\ p,q\geq 0}} 2(n+1)B^{(p)}(X)B^{(q)}(Y)Z$$
$$= \Omega(X, Y)\rho^{(k)} Z + \Omega(X, \rho^{(k)} Y)Z + \Omega(X, Z)\rho^{(k)} Y + \Omega(X, \rho^{(k)} Z)Y. \tag{10}$$

Choosing dual bases X^i, X_i for $\mathbb{R}^{2n}$ with $\Omega(X^i, X_j) = \delta^i_j$ then

$$\rho^t = \sum_i B^t(X^i)B^t(X_i),$$

i.e.

$$\rho^{(k)} = \sum_{p+q=k} \sum_i B^{(p)}(X^i)B^{(q)}(X_i).$$

If we multiply (10) by $B^{(k')}(X^i)$, set $X = X_i$ and sum over i and over $k, k' \geq 0$ so that $k + k' = K$ we get

$$(n+1) \sum_{\substack{q'+q=K \\ q,q' \geq 0}} \rho^{(q')} B^{(q)}(Y)Z = \sum_{\substack{k'+k=K \\ k',k' \geq 0}} (-B^{(k')}(Y)\rho^{(k)}Z - B^{(k')}(Z)\rho^{(k)}Y).$$

This can be written in terms of formal series

$$(n+1)\rho^t B^t(Y)Z = -B^t(Y)\rho^t Z - B^t(Z)\rho^t Y.$$

Alternatively, we may substitute $B^{(s)}(X_i)Z$ for Z in (10), set $Y = X^i$ and sum to give

$$(n+1)B^t(X)\rho^t Z = -\rho^t B^t(Z)X + B^t(Z)\rho^t X.$$

Adding the two equations after setting $X = Y$, we see that $\rho^t B^t(X) = -B^t(X)\rho^t$, so $(n-1)\rho^t B^t(X) = 0$ and, if $2n \geq 4$, $\rho^t B^t(X) = B^t(X)\rho^t = 0$ thus $(\rho^t)^2 = 0$. This in turn implies $r^t = 0$, hence $R^t = 0$ and ∇ is flat. Putting $\rho^t = 0$ in (10) yields $B^t(X)B^t(Y) = 0$. □

4. Curves of Ricci-type connections on the torus

Consider the torus T^{2n} endowed with a T^{2n}-invariant symplectic structure ω. Let ∇^0 be the standard flat, T^{2n}-invariant symplectic connection on (T^{2n}, ω). Let

$$\nabla^t = \nabla^0 + \sum_{k=1}^{\infty} t^k A^{(k)}$$

be a formal curve of symplectic connections such that $W(t) = 0$. We denote as before (2) by $\underline{A}^{(k)}$ the corresponding symmetric 3-tensors ($\underline{A}^{(k)}(X, Y, Z) = \omega(A^{(k)}(X)Y, Z)$).

We consider, as given by Lemma 8, the corresponding formal curve of 1-forms $u^t = \sum_{k=0}^{\infty} t^k u^{(k)}$ and the formal curve of functions $b^t = \sum_{k=0}^{\infty} t^k b^{(k)}$; clearly $u^{(0)} = 0$ and $b^{(0)} = 0$ since $r^{\nabla^0} = 0$.

Lemma 10. *If $\nabla^t = \nabla^0 + \sum_{k=1}^{\infty} t^k A^{(k)}$ is a formal curve of symplectic connections such that $W(t) = 0$, then the formal curvature vanishes at order 1 in t (i.e. one has $b^{(1)} = 0$, $u^{(1)} = 0$, $r^{(1)} = 0$, $R^{(1)} = 0$). Furthermore, there exists a function $U^{(1)}$ and a completely symmetric, T^{2n}-invariant 3-tensor $Q^{(1)}$ on T^{2n} such that*

$$\underline{A}^{(1)} = (\nabla^0)^3 U^{(1)} + Q^{(1)}.$$

Proof. Denote by x^a $(1 \leq a \leq 2n)$ the standard angle variables on T^{2n} and by ∂_a the corresponding T^{2n}-invariant vector fields on T^{2n} (the standard flat connection is defined by $\nabla^0_{\partial_a}\partial_b = 0$).

At order 1, since $b^{(0)} = 0$, $u^{(0)} = 0$, $r^0 = 0$, we have

(1) $db^{(1)} = 0$ by (9), so $b^{(1)}$ is a constant;
(2) $du^{(1)} = b^{(1)}\omega$ by (8); but ω is not exact by compactness of T^{2n} so $b^{(1)} = 0$ and $\nabla^0 u^{(1)} = 0$ thus $u^{(1)}(X)$ is a constant for any T^{2n}-invariant vector field X on T^{2n};
(3) Eq. (7) at order 1 yields $(\nabla^0 r^1)$ as a combination of products of ω and u^1 so that $\partial_a(r^{(1)}(\partial_b, \partial_c))$ is a constant; the periodicity of the angles x^a implies then that $\partial_a(r^{(1)}(\partial_b, \partial_c)) = 0$ so $u^{(1)} = 0$ and $r^{(1)}(\partial_b, \partial_c) = a^{(1)}_{ab}$ is a constant.

The definition of the (formal) Ricci tensor (5) yields $a^{(1)}_{ab} = -\partial_q A^{(1)q}{}_{ab}$ at order 1, hence, for each value of the indices a, b, the $2n$-form $a^{(1)}_{ab}\omega^n$ is exact; this implies

$$a^{(1)}_{ab} = 0 \quad \text{so} \quad r^{(1)} = 0 \quad \text{and thus} \quad R^{(1)} = 0.$$

The definition of the (formal) curvature tensor (4) at order 1 gives $R^{(1)}_{abcd} = \partial_a \underline{A}^{(1)}_{bcd} - \partial_b \underline{A}^{(1)}_{acd}$. Hence, for each value of the indices c, d the 1-form $\underline{A}^{(1)}_{\cdot cd}$ is closed, so there exist functions k_{cd} on T^{2n} and constants $Q^{(1)}_{bcd}$ such that

$$\underline{A}^{(1)}_{bcd} = \partial_b k^{(1)}_{cd} + Q^{(1)}_{bcd}.$$

Since ∇^t is symplectic, $\underline{A}^{(1)}_{bcd}$ is totally symmetric; the fact that $\underline{A}^{(1)}_{bcd} - \underline{A}^{(1)}_{cbd} = 0$ implies

$$\partial_b k^{(1)}_{cd} - \partial_c k^{(1)}_{bd} = -Q^{(1)}_{bcd} + Q^{(1)}_{cbd}.$$

When d is fixed, the left-hand side is an exact 2-form. The right-hand side is T^{2n}-invariant. Since there are no non-zero exact T^{2n}-invariant forms, this implies

$$Q^{(1)}_{bcd} = Q^{(1)}_{cbd}, \qquad \partial_b k^{(1)}_{cd} - \partial_c k^{(1)}_{bd} = 0.$$

Similarly, $\underline{A}^{(1)}_{bcd} - \underline{A}^{(1)}_{bdc} = 0$ gives

$$\partial_b k^{(1)}_{cd} - \partial_b k^{(1)}_{dc} = -Q^{(1)}_{bcd} + Q^{(1)}_{bdc}.$$

In this case, when c and d are fixed, the left-hand side is an exact 1-form, while the right-hand side is T^{2n}-invariant. For the same reason as above, we deduce that both members vanish:

$$Q^{(1)}_{bcd} = Q^{(1)}_{bdc}, \qquad k^{(1)}_{cd} - k^{(1)}_{dc} = \text{constant}.$$

Hence $Q^{(1)}_{bcd}$ is completely symmetric. Furthermore, for each fixed index d, the 1-form $k^{(1)}_{\cdot d}$ is closed. Hence there exist functions $S^{(1)}_d$ and constants T_{cd} such that

$$k^{(1)}_{cd} = \partial_c S^{(1)}_d + T^{(1)}_{cd}.$$

The fact that $k^{(1)}_{cd} - k^{(1)}_{dc}$ is a constant implies for the 1-form $S^{(1)}_{\cdot}$ that $dS^{(1)}$ is T^{2n}-invariant, thus $S^{(1)}$ is closed. Hence there exists a function $U^{(1)}$ and constants $V^{(1)}_d$ such that

$$S^{(1)}_d = \partial_d U^{(1)} + V^{(1)}_d.$$

Substituting, we have

$$\underline{A}^{(1)}_{bcd} = \partial^3_{bcd} U^{(1)} + Q^{(1)}_{bcd}. \qquad \square$$

Lemma 11. *If $\nabla^t = \nabla^0 + \sum_{k=1}^{\infty} t^k A^{(k)}$ is a formal curve of symplectic connections such that $W(t) = 0$, then the curvature vanishes at order 2 in t (i.e. $b^{(2)} = 0$, $u^{(2)} = 0$, $r^{(2)} = 0$, $R^{(2)} = 0$).*

Writing $\underline{A}^{(1)} = (\nabla^0)^3 U^{(1)} + Q^{(1)}$ as in Lemma 10, the formula $\nabla'^t = \nabla^0 + t\bar{Q}^{(1)}$, where $\omega(\bar{Q}^{(1)}(X)Y, Z) = Q^{(1)}(X, Y, Z)$, defines a curve of invariant flat symplectic connections on (T^{2n}, ω).

Furthermore, there exist a function $U^{(2)}$ and a T^{2n}-invariant, completely symmetric tensor $Q^{(2)}$ such that

$$\underline{A}^{(2)}_{bcd} = \underset{bcd}{\oplus} U^{(1)p}{}_b(Q^{(1)}_{pcd} + \tfrac{1}{2}U^{(1)}_{pcd}) + \tfrac{1}{2}U^{(1)p}U^{(1)}_{pbcd} + \partial^3_{bcd}U^{(2)} + Q^{(2)}_{bcd},$$

where

$$U^{(1)}_{p_1,\ldots,p_k} = \partial^k_{p_1,\ldots,p_k}U^{(1)}, \qquad U^{(1)p}{}_{q_1,\ldots,q_k} = \partial^{k+1}_{q_1,\ldots,q_k}U^{(1)}\omega^{qp}, \qquad \omega^{pq}\omega_{ql} = \delta^p_l.$$

Proof. At order 2, since $b^{(0)} = b^{(1)} = 0$, $u^{(0)} = u^{(1)} = 0$, $r^{(0)} = r^{(1)} = 0$

(1) $db^{(2)} = 0$ by (9), so $b^{(2)}$ is a constant;
(2) $du^{(2)} = b^{(2)}\omega$ by (8); so $b^{(2)} = 0$ and $\nabla^0 u^{(2)} = 0$;
(3) Eq. (7) at order 2 yields that $\partial_a(r^{(2)}(\partial_b, \partial_c))$ is a constant; again this implies $u^{(2)} = 0$ and $r^{(2)}(\partial_b, \partial_c) = a^{(2)}_{ab}$ is a constant.

The definition of the (formal) Ricci tensor yields $a^{(2)}_{ab} = -\partial_q A^{(2)q}{}_{ab} + A^{(1)p}{}_{qb}A^{(1)q}{}_{ap}$; using Lemma 10 with $Q^{(1)p}{}_{qb} = Q^{(1)}_{qbk}\omega^{kp}$:

$$\begin{aligned} A^{(1)p}{}_{qb}A^{(1)q}{}_{ap} &= Q^{(1)p}{}_{qb}Q^{(1)q}{}_{ap} + \partial_q(Q^{(1)q}{}_{ap}U^{(1)p}{}_b) + \partial_p(U^{(1)q}{}_a Q^{(1)p}{}_{qb}) \\ &\quad + \partial_q(U^{(1)p}{}_b U^{(1)q}{}_{ap}). \end{aligned}$$

Hence

$$\begin{aligned} a^{(2)}_{ab} &= Q^{(1)p}{}_{qb}Q^{(1)q}{}_{ap} - \partial_q(A^{(2)q}{}_{ab} - U^{(1)p}{}_b Q^{(1)q}{}_{ap} \\ &\quad - U^{(1)p}{}_a Q^{(1)q}{}_{pb} - U^{(1)p}{}_b U^{(1)q}{}_{ap}). \end{aligned}$$

Since there are no exact, non-zero, T^{2n}-invariant $2n$-forms on T^{2n}, we have

$$\begin{aligned} &a^{(2)}_{ab} = Q^{(1)p}{}_{qb}Q^{(1)q}{}_{ap}, \\ &\partial_q(A^{(2)q}{}_{ab} - U^{(1)p}{}_b Q^{(1)q}{}_{ap} - U^{(1)p}{}_a Q^{(1)q}{}_{pb} - U^{(1)p}{}_b U^{(1)q}{}_{ap}) = 0. \end{aligned}$$

The definition of the (formal) curvature tensor at order 2 gives $R^{(2)}_{abcd} = \partial_a\underline{A}^{(2)}_{bcd} - \partial_b\underline{A}^{(2)}_{acd} + A^{(1)p}{}_{bc}\,\underline{A}^{(1)}_{apd} - A^{(1)p}{}_{ac}\underline{A}^{(1)}_{bpd}$. Using Lemma 10 we get

$$\begin{aligned} R^{(2)}_{abcd} &= \partial_a(\underline{A}^{(2)}_{bcd} + U^{(1)}{}_{pd}Q^{(1)p}{}_{bc} - U^{(1)p}{}_c Q^{(1)}{}_{bpd} - U^{(1)p}{}_c U^{(1)}{}_{bpd}) \\ &\quad - \partial_b(\underline{A}^{(2)}_{acd} + U^{(1)}{}_{pd}Q^{(1)p}{}_{ac} - U^{(1)p}{}_c Q^{(1)}{}_{apd} - U^{(1)p}{}_c U^{(1)}{}_{apd}) \\ &\quad + Q^{(1)p}{}_{bc}Q^{(1)}{}_{apd} - Q^{(1)p}{}_{ac}Q^{(1)}{}_{bpd}. \end{aligned}$$

The $W^{(2)} = 0$ condition says that

$$R^{(2)}_{abcd} = -\frac{1}{2(n+1)}[2\omega_{ab}a^{(2)}_{cd} + \omega_{ac}a^{(2)}_{bd} + \omega_{ad}a^{(2)}_{bc} - \omega_{bc}a^{(2)}_{ad} - \omega_{bd}a^{(2)}_{ac}].$$

The fact that there does not exist a non-zero T^{2n}-invariant exact 2-form implies on the one hand

$$\partial_a(\underline{A}^{(2)}_{bcd} + U^{(1)}{}_{pd}Q^{(1)p}{}_{bc} - U^{(1)p}{}_{c}Q^{(1)}{}_{bpd} - U^{(1)p}{}_{c}U^{(1)}{}_{bpd})$$
$$-\partial_b(\underline{A}^{(2)}_{acd} + U^{(1)}{}_{pd}Q^{(1)p}{}_{ac} - U^{(1)p}{}_{c}Q^{(1)}{}_{apd} - U^{(1)p}{}_{c}U^{(1)}{}_{apd}) = 0,$$

and on the other hand

$$Q^{(1)p}{}_{bc}Q^{(1)}{}_{apd} - Q^{(1)p}{}_{ac}Q^{(1)}{}_{bpd}$$
$$= -\frac{1}{2(n+1)}[2\omega_{ab}a^{(2)}_{cd} + \omega_{ac}a^{(2)}_{bd} + \omega_{ad}a^{(2)}_{bc} - \omega_{bc}a^{(2)}_{ad} - \omega_{bd}a^{(2)}_{ac}],$$

where $a^{(2)}_{ab} = Q^{(1)p}{}_{qb}Q^{(1)q}{}_{ap}$.

This last relation tells us that the T^{2n}-invariant connection defined by $\nabla^0 + tQ^{(1)}$ (which is symplectic because of the complete symmetry) has a W tensor which is zero. Lifting everything to $\mathbb{R}^{2n}$ and applying Proposition 4 we get that the corresponding curvature vanishes identically. Hence

$$a^{(2)}_{ab} = 0, \qquad Q^{(1)p}{}_{bc}Q^{(1)}{}_{apd} - Q^{(1)p}{}_{ac}Q^{(1)}{}_{bpd} = 0.$$

This in turn implies

$$r^{(2)} = 0, \qquad R^{(2)} = 0.$$

The first relation tells us that there exist functions $k'^{(2)}_{cd}$ and constants $Q^{(2)}{}_{bcd}$ such that

$$\underline{A}^{(2)}_{bcd} - U^{(1)p}{}_{c}Q^{(1)}{}_{bpd} - U^{(1)p}{}_{d}Q^{(1)}{}_{bpc} - U^{(1)p}{}_{c}U^{(1)}{}_{bpd} = \partial_b k'^{(2)}_{cd} + Q^{(2)}{}_{bcd}.$$

This can be rewritten as

$$\underline{A}^{(2)}_{bcd} - \underset{bcd}{\circlearrowright\!\!\oplus} U^{(1)p}{}_{b}(Q^{(1)}{}_{pcd} + \tfrac{1}{2}U^{(1)}{}_{pcd}) - \tfrac{1}{2}U^{(1)p}U^{(1)}{}_{pbcd} - = \partial_b k^{(2)}_{cd} + Q^{(2)}{}_{bcd} \tag{11}$$

with

$$k^{(2)}_{cd} = k'^{(2)}_{cd} - U^{(1)p}Q^{(1)}{}_{pcd} + \tfrac{1}{2}U^{(1)p}{}_{c}U^{(1)}{}_{pd} - \tfrac{1}{2}U^{(1)p}U^{(1)}{}_{pcd}.$$

Indeed we have

$$U^{(1)p}{}_{c}U^{(1)}{}_{bpd} = \tfrac{1}{2}U^{(1)p}{}_{c}U^{(1)}{}_{bpd} + \tfrac{1}{2}\partial_b(U^{(1)p}{}_{c}U^{(1)}{}_{pd}) + \tfrac{1}{2}U^{(1)p}{}_{d}U^{(1)}{}_{bpc}$$

and also

$$\tfrac{1}{2}U^{(1)p}{}_{b}U^{(1)}{}_{cpd} = \tfrac{1}{2}\partial_b(U^{(1)p}U^{(1)}{}_{cpd}) - \tfrac{1}{2}U^{(1)p}\partial_b U^{(1)}{}_{cpd}.$$

Now the left-hand side of the Eq. (11) is totally symmetric in its indices (bcd) so the same reasoning as in Lemma 10 shows that $Q^{(2)}$ is totally symmetric and there exists a function $U^{(2)}$ so that $\partial_b k^{(2)}_{cd} = \partial^3_{bcd} U^{(2)}$. Substituting, we find

$$\underline{A}^{(2)}_{bcd} = \underset{bcd}{\circlearrowleft\!\!\!\!\bigoplus} U^{(1)p}{}_b (Q^{(1)}{}_{pcd} + \tfrac{1}{2} U^{(1)}{}_{pcd}) + \tfrac{1}{2} U^{(1)p} U^{(1)}{}_{pbcd} + \partial^3_{bcd} U^{(2)} + Q^{(2)}_{bcd}$$

which ends the proof of the lemma. □

5. A recurrence lemma

Lemma 12. *Let ∇^t be a formal curve of symplectic connections on (T^{2n}, ω) such that $\nabla^{(0)} = \nabla^0$, and $W^t = 0$. Assume that, for all orders $l < k$, $\underline{A}^{(l)}$, and thus $r^{(l)}$, $u^{(l)}$, $b^{(l)}$ are T^{2n}-invariant. Then, at order k, $r^{(k)}$, $u^{(k)}$, $b^{(k)}$ are T^{2n}-invariant, and there exist a function $U^{(k)}$ on T^{2n} and a T^{2n}-invariant completely symmetric 3-tensor $Q^{(k)}$ such that*

$$\underline{A}^{(k)} = \partial^3 U^{(k)} + Q^{(k)}.$$

Proof. Assume that, up to order $k-1$ (included), $\underline{A}^{(l)}_{abc}$, $r^{(l)}_{ab}$, $u^{(l)}_a$, $b^{(l)}$ are T^{2n}-invariant. Then, at order k, we have

(i) $$R^{(k)}_{abcd} = \partial_a \underline{A}^{(k)}_{bcd} - \partial_b \underline{A}^{(k)}_{acd} + \sum_{\substack{s+s'=k\\ s,s'>0}} A^{(s)p}_{bc} \underline{A}^{(s')}{}_{apd} - A^{(s)p}_{ac} \underline{A}^{(s')}{}_{bpd};$$

(ii) $$r^{(k)}_{ac} = -\partial_q A^{(k)q}_{ac} + \sum_{\substack{s+s'=k\\ s,s'>0}} A^{(s)p}_{qc} A^{(s')q}_{ap};$$

(iii) $$\partial_c r^{(k)}_{ab} - \sum_{\substack{s+s'=k\\ s,s'>0}} A^{(s)p}_{ca} r^{(s')}_{pb} + \Gamma^{(s)p}_{cb} r^{(s')}_{ap} = \frac{1}{2n+1} (\omega_{cb} u^{(k)}_a + \omega_{ca} u^{(k)}_b);$$

(iv) $$\partial_b u^{(k)}_a - \sum_{\substack{s+s'=k\\ s,s'>0}} A^{(s)p}_{ba} u^{(s')}_p = -\frac{1+2n}{2(1+n)} \sum_{\substack{s+s'=k\\ s,s'>0}} r^{(s)}_{bc} r^{(s')c}_a + b^{(k)} \omega_{ba};$$

(v) $$\partial_a b^{(k)} = \frac{1}{1+n} \sum_{\substack{s+s'=k\\ s,s'>0}} \bar{u}^{(s)c} r^{(s')}_{ca}.$$

Relation (v) implies that $\mathrm{d}b^{(k)}$ is T^{2n}-invariant. Hence $\mathrm{d}b^{(k)} = 0$ and $b^{(k)}$ is a constant. Antisymmetrising (iv) we get that $\mathrm{d}u^{(k)} - b^{(k)}\omega$ is a T^{2n}-invariant 2-form, hence $\mathrm{d}u^{(k)} = 0$ and

$$b^{(k)} \omega_{ba} - \frac{1+2n}{2(1+n)} \sum_{\substack{s+s'=k\\ s,s'>0}} r^{(s)}_{bc} r'^{(s)c}{}_a = 0.$$

Also

$$\partial_b u^{(k)}_a = \sum_{\substack{s+s'=k\\ s,s'>0}} A^{(s)p}_{ba} u'^{(s)}_p.$$

Using periodicity again and the fact that the right-hand side is a constant, we see that the $u_a^{(k)}$ are constants. Relation (iii) tells us, for the same reason, that the $r_{ab}^{(k)}$ are constants. Finally, from (i) and the $W^t = 0$ condition, we get that $\partial_a \underline{A}_{bcd}^{(k)} - \partial_b \underline{A}_{acd}^{(k)}$ is a constant hence

$$\partial_a \underline{A}_{bcd}^{(k)} - \partial_b \underline{A}_{acd}^{(k)} = 0. \tag{1}$$

The reasoning of Lemma 10 applies to Eq. (1) so there exist a function $U^{(k)}$ on T^{2n} and a T^{2n}-invariant completely symmetric 3 tensor $Q^{(k)}$ such that

$$\underline{A}^{(k)} = \partial^3 U^{(k)} + Q^{(k)}. \qquad \square$$

We can now proceed to the proof of the main theorem.

Theorem 13. *Let ∇^t be a formal curve of symplectic connections on (T^{2n}, ω) with ∇^0 the standard connection, and $W^t = 0$. Then there exists a formal curve of symplectomorphisms ψ_t such that $\tilde{\nabla}^t := \psi_t \cdot \nabla^t$ is a formal curve of symplectic connections which is T^{2n}-invariant and has $\tilde{W}^t = 0$, hence is flat. In particular, ∇^t is flat.*

Proof. If $\nabla^t = \nabla^0 + \sum_{k=0}^{\infty} t^p A^{(p)}$ is any formal curve of symplectic connections, one defines as in (3) the action of a formal curve ψ_t of symplectomorphisms on ∇^t:

$$(\psi_t \cdot \nabla^t)_X Y = \psi_t \cdot (\nabla^t_{\psi_t^{-1} \cdot X} \psi_t^{-1} \cdot Y).$$

Consider a formal 1-parameter group $\psi_f(t)$ of symplectomorphisms generated by a hamiltonian vector field X_f ($i(X_f)\omega = \mathrm{d}f$) and consider the formal curve of symplectomorphisms defined by $\psi_f^k(t) = \psi_f(t^k)$. Write

$$\psi_f^k(t) \cdot \nabla^t = \nabla^0 + \sum_{p=0}^{\infty} t^p \tilde{A}^{(p)}$$

then $\tilde{A}^{(p)} = A^{(p)}, \forall p < k$ and

$$\tilde{A}_X^{(k)} Y = A_X^{(k)} Y + [X_f, \nabla_Y^0 Z] - \nabla^0_{[X_f, Y]} Z - \nabla_Y^0 [X_f, Z].$$

Observe that

$$[X_f, \nabla_Y^0 Z] - \nabla^0_{[X_f, Y]} Z - \nabla_Y^0 [X_f, Z] = R^0(X_f, Y)Z + ((\nabla^0)^2 X_f)(Y, Z)$$

and $\omega(((\nabla^0)^2 X_f)(Y, Z), T) = ((\nabla^0)^3 f)(Y, Z, T)$.

Assume now that the curve $\nabla_t = \nabla^0 + \sum_{k=0}^{\infty} t^p A^{(p)}$ is a curve of symplectic connections on the torus (T^{2n}, ω) and that ∇^0 is the standard flat connection.

At order 1, we have seen in Lemma 10 that $\underline{A}^{(1)} = (\nabla^0)^3 U^{(1)} + Q^{(1)}$ so choosing $f_1 = -U^{(1)}$ and $\psi^{(1)}(t) = \psi_{f_1}(t)$ as defined above we see that

$$\psi^{(1)}(t) \cdot \nabla^t = \nabla^0 + t\bar{Q}^{(1)} + \sum_{p=2}^{\infty} t^p \tilde{A}^{(p)}$$

with $\omega(\bar{Q}^{(1)}(X)Y, Z) = Q^{(1)}(X, Y, Z)$.

Assume now that one has found a formal curve $\psi^{(k-1)}(t)$ of symplectomorphisms so that

$$\psi^{(k-1)}(t)\cdot\nabla^t=\nabla^0+\sum_{p=1}^{k-1}t^p\bar{Q}^{(p)}+\sum_{p=k}^{\infty}t^p\tilde{A}^{(p)},$$

where the $\bar{Q}^{(p)}$ are T^{2n}-invariant.

At order k, we have seen in Lemma 12 that $\underline{A}^{(k)}=(\nabla^0)^3U^{(k)}+Q^{(k)}$ where $Q^{(k)}$ is T^{2n}-invariant, so choosing $f_k=-U^{(k)}$, $\psi^k_{f_k}(t)$ as defined above and $\psi^{(k)}(t)=\psi_{f_k}(t^k)\circ\psi^{(k-1)}(t)$ we see that

$$\psi^{(k)}(t)\cdot\nabla^t=\psi_{f_k}(t^k)\cdot\psi^{(k-1)}(t)\cdot\nabla^t=\nabla^0+\sum_{p=1}^{k}t^p\bar{Q}^{(p)}+\sum_{p=k+1}^{\infty}t^p\tilde{A}^{(p)}$$

with $\omega(\bar{Q}^{(k)}(X)Y,Z)=Q^{(k)}(X,Y,Z)$. By induction this proves that one can build a formal curve of symplectomorphisms

$$\psi(t)=\cdots\circ\psi_{(f_k)}(t^k)\circ\cdots\circ\psi_{f_2}(t^2)\circ\psi_{f_1}(t),$$

so that $\tilde{\nabla}(t):=\psi(t)\cdot\nabla(t)$ is a formal curve of symplectic connections which is T^{2n}-invariant and has $\tilde{W}(t)=0$. Lifting the connection to $\mathbb{R}^{2n}$ and using Lemma 9 shows that $\tilde{\nabla}(t)$ has vanishing curvature. Since $\nabla(t)=(\psi(t))^{-1}\cdot\tilde{\nabla}(t)$, its curvature is 0 so $\nabla(t)$ is flat. □

The above theorem implies the following:

Theorem 14. *Let ∇^t be an analytic curve of analytic symplectic connections on (T^{2n},ω) such that ∇^0 is the standard flat connection on T^{2n}, and such that $W^t=0$. Then the curvature R^t of ∇^t vanishes.*

6. Equivalence of formal curves of connections

In this section we study the question of when two formal curves of flat invariant connections on T^{2n} are equivalent by a formal curve of symplectomorphisms. First we consider the question on $(\mathbb{R}^{2n},\Omega)$. Here it is easy to answer.

The first case to consider is the case of a single flat invariant connection $\nabla^A=\nabla^0+A$ on $(\mathbb{R}^{2n},\Omega)$. We have seen that such a connection is given by a linear map $A:\mathbb{R}^{2n}\to\mathfrak{sp}(2n,\mathbb{R})$ satisfying $A(X)A(Y)=0$ and $\Omega(A(X)Y,Z)$ completely symmetric. Define $\psi^A:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ by

$$\psi^A(x)=x-\tfrac{1}{2}A(x)x.$$

Proposition 15. *ψ^A is a symplectomorphism of $(\mathbb{R}^{2n},\Omega)$ satisfying $\psi^A\cdot\nabla^0=\nabla^A$.*

Proof. It is enough to check that ψ^A is a symplectomorphism on constant vector fields. We make extensive use of the fact that $A(X)A(Y)=0$. If X is a constant vector field then

$$\psi^A_*X_x=\left.\frac{\mathrm{d}}{\mathrm{d}t}\psi^A(x+tX)\right|_{t=0}=(X-A(x)X)_{\psi^A(x)},$$

thus $\psi^A \cdot X = X - A(\cdot)X$. Hence

$$\Omega(\psi^A \cdot X, \psi^A \cdot Y)(x) = \Omega(X - A(x)X, Y - A(x)Y) = \Omega(X, Y).$$

It is easy to see that ψ^{-A} is an inverse for ψ^A so that ψ^A is a symplectomorphism. Indeed, $t \mapsto \psi^{tA}$ is a 1-parameter group of symplectomorphisms with generator the symplectic vector field $(X_A)_x = -\frac{1}{2}A(x)x_x$.

Finally, for constant vector fields X, Y

$$(\psi^A \cdot \nabla^0)_X Y = \psi^A \cdot (\nabla^0_{\psi^{-A}\cdot X}\psi^{-A} \cdot Y) = \psi^A \cdot ((X + A(\cdot)X)(A(\cdot)Y)).$$

But

$$(X + A(\cdot)X)(A(\cdot)Y)_x = \frac{\mathrm{d}}{\mathrm{d}t}A(x + t(X + A(x)X))Y\Big|_{t=0} = A(X)Y$$

so

$$(\psi^A \cdot \nabla^0)_X Y = \psi^A \cdot (A(X)Y) = A(X)Y = \nabla^A_X Y. \qquad \square$$

If $\nabla^t = \nabla^0 + A^t$ is a formal curve of invariant flat connections on $(\mathbb{R}^{2n}, \Omega)$ given by a curve of linear maps $A^t : \mathbb{R}^{2n} \to \mathfrak{sp}(2n, \mathbb{R})[[t]]$ satisfying $A^t(X)A^t(Y) = 0$ and $\Omega(A^t(X)Y, Z)$ completely symmetric, we define a formal curve of vector fields X_{A^t} by

$$X_{A^t}(f)(x) = -\tfrac{1}{2}(A_t(x)x)_x f$$

and set

$$\psi_{A^t} = \exp X_{A^t}.$$

Proposition 16. *ψ_{A^t} is a formal curve of symplectomorphisms of $(\mathbb{R}^{2n}, \Omega)$ and $\psi_{A^t} \cdot \nabla^0 = \nabla^{A^t}$.*

Proof. As the exponential of a derivation, ψ_{A^t} is invertible with inverse $\exp(-X_{A^t}) = \psi_{-A^t}$. Moreover, $\psi_{A^t} \cdot X = \exp \operatorname{ad} X_{A^t} X$ and it is easy to verify that $\operatorname{ad} X_{A^t} X = A^t(\cdot)X$, $(\operatorname{ad} X_{A^t})^2 X = 0$ so that $\psi_{A^t} \cdot X = X - A^t(\cdot)X$ as before. Likewise $\psi_{-A^t} \cdot X = X + A^t(\cdot)X$ so that

$$(\psi_{A^t} \cdot \nabla^0)_X Y = \psi_{A^t} \cdot (\nabla^0_{\psi_{-A^t}\cdot X}(Y + A^t(\cdot)Y)) = A^t(X)Y. \qquad \square$$

In particular the above proves.

Theorem 17. *For two curves $\tilde{\nabla}^t$ and $\tilde{\nabla}'^t$ of invariant flat connections of Ricci-type on $(\mathbb{R}^{2n}, \Omega)$ with $\tilde{\nabla}^0 = \tilde{\nabla}'^0$ the trivial connection, there always exists a formal curve of symplectomorphisms $\widetilde{\psi_t}$ so that $\tilde{\psi}_t \cdot \tilde{\nabla}^t = \tilde{\nabla}'^t$.*

Finally, we need to know what is the general form of a formal curve of symplectomorphisms of $(\mathbb{R}^{2n}, \Omega)$ which fixes the trivial connection ∇^0.

Proposition 18. *Let $\psi_t = \sigma^* \circ \exp X_t$ be a formal curve of symplectomorphisms with $\psi_t \cdot \nabla^0 = \nabla^0$ then $\sigma(x) = Cx + d$ and $(X_t)_x = (C_t(x) + d_t)_x$ where $C \in Sp(2n, \mathbb{R})$, $d \in \mathbb{R}^{2n}$, $C_t \in t\mathfrak{sp}(2n, \mathbb{R})[\![t]\!]$ and $d_t \in t\mathbb{R}^{2n}[\![t]\!]$.*

Proof. Evaluation at $t = 0$ shows that $\sigma \cdot \nabla^0 = \nabla^0$ so that $\sigma(x) = Cx + d$ where $C \in Sp(2n, \mathbb{R})$ and $d \in \mathbb{R}^{2n}$. Hence $\exp X_t \cdot \nabla^0 = \nabla^0$. ∇^0 is the connection for which constant vector fields are parallel, so $(\exp X_t \cdot \nabla^0)_X Y = 0$ for constant vector fields X, Y. Hence $\nabla^0_{\exp(-X_t)\cdot X} \exp(-X_t)\cdot Y = 0$ and so $\nabla^0_X \exp(-X_t)\cdot Y = 0$. But the only parallel vector fields for ∇^0 are the constant fields, so $\exp(-X_t) \cdot Y$ is constant. The leading term is $-t[X^{(1)}, Y]$ and hence $[X^{(1)}, Y]$ is constant. Since $X^{(1)}$ is symplectic, this means $X^{(1)}_x = (C_1 x + d_1)_x$ where $C_1 \in \mathfrak{sp}(2n, \mathbb{R})$. Further $\exp tX^{(1)}$ preserves ∇^0 and $\exp(-tX(1)) \circ \exp X_t = \exp X'_t$ with $X'_t = O(t^2)$ so we can recurse to conclude that $(X_t)_x = (C_t(x) + d_t)_x$ for formal curves $C_t \in t\mathfrak{sp}(2n, \mathbb{R})[\![t]\!]$ and $d_t \in t\mathbb{R}^{2n}[\![t]\!]$. □

Theorem 19. *Let ∇^t and ∇'^t be two curves of invariant flat connections on T^{2n} with $\nabla^0 = \nabla'^0$ the trivial connection and suppose that there is a formal curve of symplectomorphisms ψ_t with $\psi_t \cdot \nabla^t = \nabla'^t$ then there is an element $C \in Sp(2n, \mathbb{Z})$ such that as a symplectomorphism of T^{2n} we have $\nabla'^t = C \cdot \nabla^t$.*

Proof. We lift the connections and ψ_t to $\mathbb{R}^{2n}$ and denote the lifts by a tilde. $\tilde{\psi}_t \cdot \tilde{\nabla}^t = \widetilde{\nabla'}^t$. Then $\tilde{\nabla}^t = \nabla^0 + A^t$, $\widetilde{\nabla'}^t = \nabla^0 + B^t$ where $A^t, B^t : \mathbb{R}^{2n} \to \mathfrak{sp}(2n, \mathbb{R})[\![t]\!]$ are linear with the usual properties. Thus

$$(\tilde{\psi}_t \circ \psi_{A^t}) \cdot \nabla^0 = \psi_{B^t} \cdot \nabla^0$$

and hence

$$\tilde{\psi}_t \circ \psi_{A^t} = \psi_{B^t} \circ \sigma^* \circ \exp X_t,$$

where $\sigma(x) = Cx + d$ and $(X_t)_x = (C_t x + d_t)_x$.

Now $\psi_{B^t} \circ \sigma^* = \sigma^* \circ \sigma^{-1^*} \circ \exp X_{B^t} \circ \sigma^* = \sigma^* \circ \exp \sigma \cdot X_{B^t}$ and

$$(\sigma \cdot X_{B^t})_x = (X_{C\cdot B^t})_x + ((C \cdot B^t)(x)d)_x - \tfrac{1}{2}((C \cdot B^t)(d)d)_x$$

whilst the last two terms are in the semidirect product $t\mathfrak{sp}(2n, \mathbb{R})[\![t]\!] + t\mathbb{R}^{2n}[\![t]\!]$ which is pronilpotent. We can exponentiate this equation in the form

$$\exp \sigma \cdot X_{B^t} = \exp X_{C\cdot B^t} \exp Z_t$$

with $Z_t \in t\mathfrak{sp}(2n, \mathbb{R})[\![t]\!] + t\mathbb{R}^{2n}[\![t]\!]$. At order zero we see that σ must be the lift of ψ^0 and so must preserve the lattice: $C \in Sp(2n, \mathbb{Z})$. Then $\sigma^{-1} \circ \tilde{\psi}_t$ descends to the torus and leads off with the identity, so is of the form $\exp L_t$ where L_t is a formal series of periodic vector fields on $\mathbb{R}^{2n}$. Thus we have, combining the terms in $\exp t\mathfrak{sp}(2n, \mathbb{R})[\![t]\!] + t\mathbb{R}^{2n}[\![t]\!]$ and renaming as Z_t,

$$\exp L_t = \exp X_{C\cdot B^t} \exp Z_t \exp(-X_{A^t}).$$

Equating the coefficient of t on both sides we see that

$$L^{(1)} = X_{C\cdot B^{(1)}} + Z^{(1)} - X_{A^{(1)}}$$

and since linear and quadratic functions are never periodic we see that $C \cdot B^{(1)} = A^{(1)}$, and $L^{(1)} = Z^{(1)}$ is constant. A simple recursion (moving constant terms past $\exp X_{C\cdot B^t}$) suffices to see that $A^t = C \cdot B^t$. □

So we have the following:

Theorem 20. *The moduli space of curves of Ricci-type symplectic connections starting with the standard flat connection on* (T^{2n}, ω) *under the action of formal curves of symplectomorphisms is described by the space of formal curves* $A^t : \mathbb{R}^{2n} \to \mathfrak{sp}(2n, \mathbb{R})[[t]]$ *satisfying* $A^t(X)A^t(Y) = 0$ *and* $A^t(X)Y = A^t(Y)X$, *modulo the action of* $Sp(2n, \mathbb{Z})$.

It is worth noting that a curve of Ricci-type connections on the torus is equivalent to the constant curve at the trivial connection when lifted to $\mathbb{R}^{2n}$.

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